

Chaotic oscillations in finite quantum systems: trapped Bose-Einstein condensates¹

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Abstract

We discuss the recently achieved Bose-Einstein condensation for alkali-metal atoms in magnetic traps. The theoretically predicted low-energy collective oscillations of the condensate have been experimentally confirmed by laser imaging techniques. We show by using Poincaré sections that at higher energies non-linear effects appear and oscillations become chaotic.

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1 Introduction

Finite many-body quantum systems, both fermionic and bosonic, exhibit collective oscillations, which are nowadays experimentally detected with sophisticated devices. In many cases, these collective oscillations can become strongly aperiodic and also chaotic [1-3].

In this short report, we analyze the collective modes of a trapped Bose condensate of alkali atoms. The Bose-Einstein condensation (BEC) is the macroscopic occupation of the ground-state of the system of bosons. From 1995 we have experimental results interpreted as an evidence of BEC in dilute vapors of confined alkali-metal atoms (⁸⁷Rb, ²³Na and ⁷Li) [4-6]. The experiments with alkali-metal atoms generally consist of a laser cooling and confinement in an external potential (a magnetic or magneto-optical trap) and an evaporative cooling (temperature of the order of 100 nK) [4-6]. Nowadays a dozen of experimental groups have achieved the BEC by using different geometries of the confining trap and atomic species.

The dynamics of the Bose condensate can be accurately described by the Gross-Pitaevskii equation [7] of mean-field approximation. The theoretically predicted low-energy collective oscillations of the condensate have been experimentally confirmed by laser imaging techniques [8]. In this short report we show that at higher energies non-linear effects appear and eventually the collective oscillations become chaotic.

2 Mean-field theory of BEC

The many-body problem of N identical bosonic atoms in an external trapping potential can be formulated within the non-relativistic quantum field theory [9-11]. The Lagrangian density of

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the Schrödinger field is given by

$$\begin{aligned} \mathcal{L} = & \hat{\psi}^+(\mathbf{r}, t) \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}, t) \\ & - \frac{1}{2} \int d^3\mathbf{r}' \hat{\psi}^+(\mathbf{r}, t) \hat{\psi}^+(\mathbf{r}', t) V(|\mathbf{r} - \mathbf{r}'|) \hat{\psi}(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) , \end{aligned} \quad (1)$$

where $U(\mathbf{r})$ is the external potential and $V(|\mathbf{r} - \mathbf{r}'|)$ is the interatomic potential. The quantum bosonic field $\hat{\psi}(\mathbf{r})$ satisfies the standard equal-time commutation rules.

The Lagrangian is invariant under the global $U(1)$ gauge transformation

$$\hat{\psi}(\mathbf{r}, t) \rightarrow e^{i\alpha} \hat{\psi}(\mathbf{r}, t) , \quad (2)$$

which implies the conservation of the total number of particles

$$\hat{N} = \int d^3\mathbf{r} \hat{\psi}^+(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) . \quad (3)$$

The Bose-Einstein condensation (BEC) is the macroscopic occupation of the N -body ground-state $|\mathbf{O}\rangle$ of the system. To study BEC a useful mean-field prescription is to separate out the condensate contribution to the bosonic field operator in the following way

$$\hat{\psi}(\mathbf{r}, t) = \phi(\mathbf{r}, t) + \hat{\Sigma}(\mathbf{r}, t) , \quad (4)$$

where $\phi(\mathbf{r}, t) = \langle \mathbf{O} | \hat{\psi}(\mathbf{r}, t) | \mathbf{O} \rangle$ is the so-called macroscopic wavefunction (or order parameter) of the condensate, and $\hat{\Sigma}(\mathbf{r}, t)$ is the fluctuation operator, such that $\hat{\Sigma}(\mathbf{r}, t) | \mathbf{O} \rangle = 0$. Note that this prescription breaks the $U(1)$ global gauge symmetry of the system [9-11].

Alkali vapors are quite dilute and at zero temperature the atoms are practically all in the condensate [4-6]. Thus we can neglect the quantum depletion due to the operator $\hat{\Sigma}$ and the macroscopic wavefunction is normalized to the total number of atoms. Moreover, the range of the atom-atom interaction $V(r)$ is believed to be short in comparison to the typical length scale of variations of atomic wave functions. The atom-atom interaction is usually replaced by an effective zero-range pseudo-potential, $V(r) = g\delta^3(r)$, where $g = 4\pi\hbar^2 a_s/m$ is the scattering amplitude and a_s is the s-wave scattering length. This scattering length is positive (repulsive interaction) for ^{87}Rb and ^{23}Na but negative (attractive interaction) for ^7Li . Within these approximations, the Lagrangian density becomes a local function of the condensate wavefunction, namely

$$\mathcal{L} = \phi^*(\mathbf{r}, t) \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) \right] \phi(\mathbf{r}, t) - \frac{1}{2} g |\phi(\mathbf{r}, t)|^4 . \quad (5)$$

By imposing the least action principle one obtains the following Euler-Lagrange equation

$$i\hbar \frac{\partial}{\partial t} \phi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + g |\phi(\mathbf{r}, t)|^2 \right] \phi(\mathbf{r}, t) , \quad (6)$$

which is a nonlinear Schrödinger equation and is called time-dependent Gross-Pitaevskii (GP) equation [7]. Note that the GP equation is nothing but the mean-field (Hartree) approximation of the exact time-dependent Schrödinger equation of the N -body problem, where the totally symmetric many-particle wavefunction Ψ of the system is decomposed in the following way $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = \phi(\mathbf{r}_1, t) \phi(\mathbf{r}_2, t) \dots \phi(\mathbf{r}_N, t)$.

3 Hydrodynamic equations of the Bose condensate

The complex macroscopic wavefunction $\phi(\mathbf{r}, t)$ of the condensate can be written in terms of a modulus and a phase, as follows

$$\phi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)} . \quad (7)$$

The phase S fixes the velocity field $\mathbf{v} = (\hbar/m)\nabla S$. The GP equation can hence be rewritten in the form of two coupled hydrodynamic equations [12] for the density and the velocity field

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (8)$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla \left(U + g\rho - \frac{\hbar^2}{2m\sqrt{\rho}}\nabla^2\sqrt{\rho} + \frac{mv^2}{2} \right) = 0 . \quad (9)$$

If the repulsive interaction among atoms is strong enough, then the density profiles become smooth and one can safely neglect the kinetic pressure term in \hbar^2 in the equation for the velocity field, which then takes the form

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla \left(U + g\rho + \frac{mv^2}{2} \right) = 0 . \quad (10)$$

In the current experiments with alkali metal-atoms, the external trap is well approximated by a harmonic potential

$$U(\mathbf{r}) = \frac{m}{2}(\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2) . \quad (11)$$

The ground-state solution ($\mathbf{v} = 0$) of Eq. (10) is given by

$$\rho(\mathbf{r}) = g^{-1} [\mu - U(\mathbf{r})] , \quad (12)$$

in the region where $\mu > U(\mathbf{r})$, and $\rho = 0$ outside. The normalization condition on $\rho(\mathbf{r})$ provides $\mu = (\hbar\omega_h/2)(15Na_s/a_h)^{2/5}$, where $\omega_h = (\omega_1\omega_2\omega_3)^{1/3}$ and $a_h = (\hbar/m\omega_h)^{1/2}$.

An analytic class of solutions [13] of the time-dependent hydrodynamic equations (8) and (10), which are valid when the condition $Na_s/a_h \gg 1$ is satisfied, is found by writing the density in the form

$$\rho(\mathbf{r}, t) = a_0(t) - a_1(t)x^2 - a_2(t)y^2 - a_3(t)z^2 \quad (13)$$

in the region where $\rho(\mathbf{r}, t)$ is positive, and the velocity field as

$$\mathbf{v}(\mathbf{r}, t) = \frac{1}{2}\nabla[b_1(t)x^2 + b_2(t)y^2 + b_3(t)z^2] . \quad (14)$$

The coefficient a_0 is fixed by the normalization of the density $a_0 = (15N/8\pi)^{2/5}(a_x a_y a_z)^{1/5}$. By inserting these expressions in the hydrodynamic equations one finds 6 coupled differential equations for the time-dependent parameters $a_i(t)$ and $b_i(t)$. By introducing new variables q_i , defined by $a_i = m\omega_i^2(2gq_i^2q_1q_2q_3)^{-1}$, the hydrodynamic equations give $a_i = \dot{q}_i/q_i$ and

$$\ddot{q}_i + \omega_i^2 q_i = \frac{\omega_i^2}{q_i q_1 q_2 q_3} , \quad (15)$$

with $i = 1, 2, 3$. The second and third terms give the effect of the external trap and of the interatomic forces, respectively. It is important to observe that, using the new variables q_i , the equations of motion do not depend on the value of the coupling constant g . In terms of q_i the mean square radii of the condensate are $\langle x_i^2 \rangle = (2\mu/m\omega_i)q_i^2$ and the velocities are $\langle v_i^2 \rangle = (2\mu/m\omega_i)\dot{q}_i^2$ [13].

4 BEC Collective Modes and Chaos

The three differential equations (15) are the classical equations of motion of a system with coordinates q_i and Lagrangian given by

$$L = \frac{1}{2}(\omega_1^{-2}\dot{q}_1^2 + \omega_2^{-2}\dot{q}_2^2 + \omega_3^{-2}\dot{q}_3^2) - \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) - \frac{1}{q_1 q_2 q_3} . \quad (16)$$

This Lagrangian describes collective modes of the Bose condensate for $Na_s/a_h \gg 1$ [13]. As stressed previously, in such a case the collective dynamics of the condensate does not depend on the number of atoms or the scattering length. The minimum of the effective potential is at $q_i = 1$, $i = 1, 2, 3$. The mass matrix M of the kinetic energy and the Hessian matrix Λ of the potential energy at the equilibrium point are given by

$$M = \begin{pmatrix} \omega_1^{-2} & 0 & 0 \\ 0 & \omega_2^{-2} & 0 \\ 0 & 0 & \omega_3^{-2} \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} . \quad (17)$$

The low-energy collective excitations of the condensate are the small oscillations of variables q_i 's around the equilibrium point. The calculation of the normal mode frequencies Ω for the motion of the condensate is reduced to the eigenvalue problem $\Lambda - \Omega^2 M = 0$, which gives

$$\Omega^6 - 3(\omega_1^2 + \omega_2^2 + \omega_3^2)\Omega^4 + 8(\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2)\Omega^2 - 20\omega_1^2\omega_2^2\omega_3^2 = 0 . \quad (18)$$

Note that this formula has been recently obtained by using a variational approach [14] and also by studying hydrodynamic density fluctuations of the condensate [15]. For an axially symmetric trap, where $\omega_1 = \omega_2 = \omega_\perp$, the previous equation gives

$$\Omega_{1,2} = \left(2 + \frac{3}{2}\lambda^2 \pm \frac{1}{2}(16 + 9\lambda^4 - 16\lambda^2)^{1/2}\right)^{1/2} \omega_\perp , \quad \Omega_3 = \sqrt{2} \omega_\perp , \quad (19)$$

where $\lambda = \omega_3/\omega_\perp$ is the asymmetry parameter of the trap [12]. Observe that the experimental results obtained on sodium vapors at MIT ($\lambda = \sqrt{8}$) are in good agreement with the theoretical values predicted by (19) [16]. In the case of an isotropic harmonic trap ($\lambda = 1$) with frequency ω , one obtains $\Omega_{1,2} = \sqrt{5} \omega$, $\Omega_3 = \sqrt{2} \omega$ [12].

In most experiments the confining trap has axial symmetry [4-6]. Let us analyze this case in detail. Because of the axial symmetry, we can impose $q_1 = q_2 = q_\perp$. Moreover, by using the adimensional time $\tau = \omega_\perp t$, the Hamiltonian of the BEC collective modes becomes

$$H = p_\perp^2 + \frac{1}{2}\lambda^2 p_3^2 + q_\perp^2 + \frac{1}{2}q_3^2 + \frac{1}{q_\perp^2 q_3} , \quad (20)$$

where $p_\perp = dq_\perp/d\tau$ and $p_3 = \lambda^{-2}dq_3/d\tau$ are the conjugate momenta. Note that the condition $q_1 = q_2$ restricts collective modes to monopole oscillations, where the third component of the angular momentum, that is a good quantum number, is zero [12]. Near the minimum of the potential the trajectories in the phase-space are periodic or quasi-periodic. On the contrary, far from the minimum, the effect of nonlinearity becomes important. As the KAM theorem [17] predicts, parts of phase space become filled with chaotic orbits, while in other parts the toroidal surfaces of the integrable system are deformed but not destroyed.

We use a symplectic Euler method (leap-Frog) to numerically compute the trajectories. The time-step is $\Delta\tau = 10^{-4}$ and the energy is conserved to the sixth digit. The conservation

of energy restricts any trajectory of the four-dimensional phase space to a three-dimensional energy shell. At a particular energy, the restriction $q_{\perp} = 1$ defines a two-dimensional surface in the phase space, which is called Poincaré section. Each time a particular trajectory passes through the surface a point is plotted at the position of intersection (q_3, p_3) . We employ a first-order interpolation process to reduce inaccuracies due to the use of a finite step length.

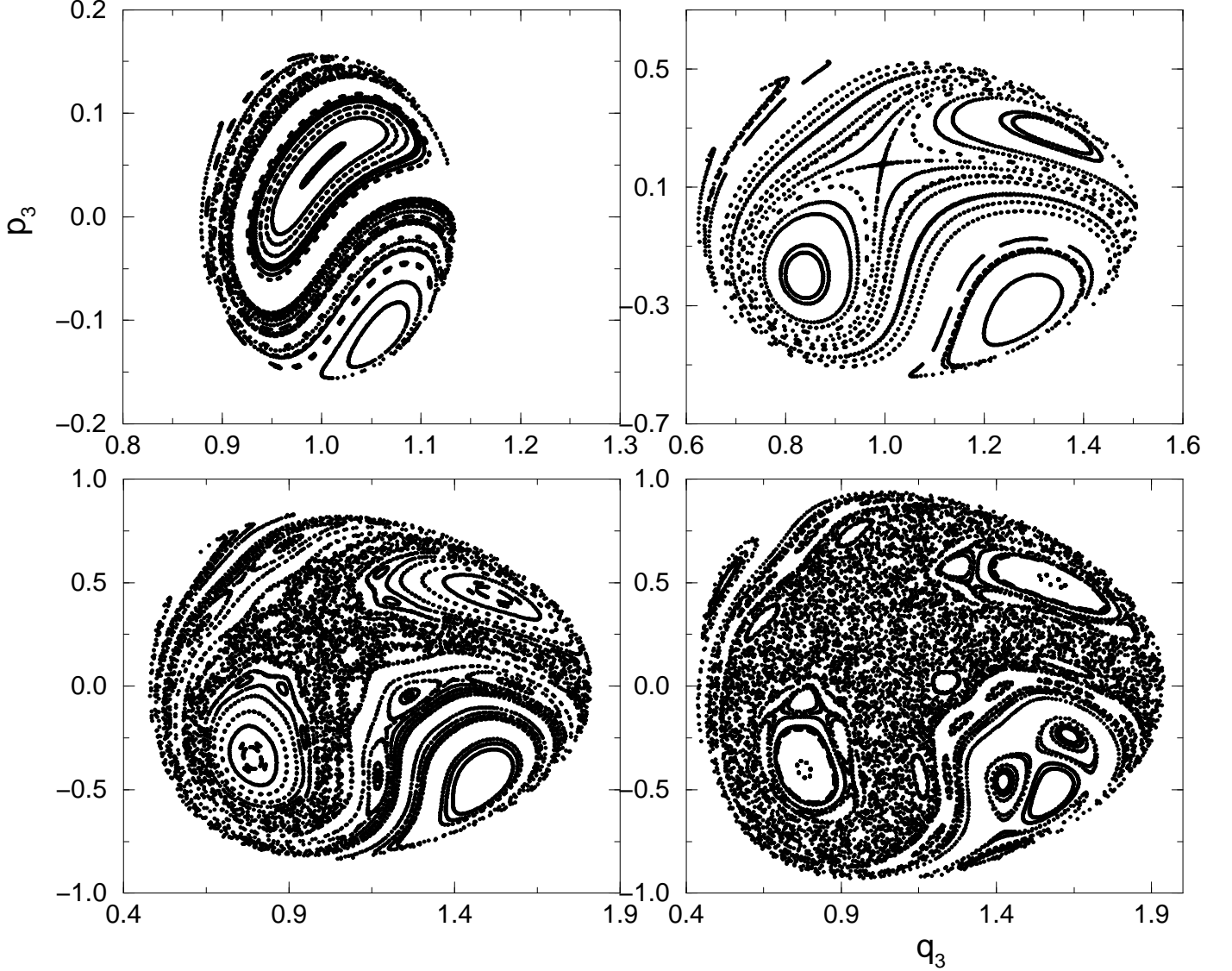


Figure 1: Poincaré sections with $\lambda = \sqrt{2}$. Each panel is at a fixed energy. From left to right and from top to bottom: $\chi = 1\%$, $\chi = 12\%$, $\chi = 28\%$, $\chi = 36\%$. χ is the relative increase of the energy with respect to the ground-state (minimum of the potential energy).

In Figure 1 we plot Poincaré sections of the system with $\lambda = \sqrt{2}$. In each panel there is a Poincaré section with a fixed value of the energy of the system. At each energy value, we have chosen different initial conditions $[q_{\perp}(0), q_3(0), p_{\perp}(0), p_3(0)]$ for the dynamics. Actually $p_{\perp}(0)$ has been fixed by the conservation of energy. Integration time is 400 in adimensional units, that is less than 1 second (the life-time of the condensate is about 10 seconds). Note that the CPU time to calculate a Poincaré section with a dozen of initial conditions is about 30 seconds. Chaotic regions on the Poincaré section are characterized by a set of randomly distributed points and regular regions by dotted or solid curves. Let χ be the relative increase of the energy with respect to the ground-state (minimum of the potential energy). For $\chi = 1\%$ and $\chi = 12\%$

the trajectories are still all regular but for $\chi = 28\%$ there is chaotic sea. For $\chi = 36\%$ most trajectories are chaotic.

It is important to observe that a strong enhancement of nonlinear effects and eventually chaos can be obtained not only by increasing the energy but also by changing the anisotropy λ of the trap. In fact, as shown in Ref. [13], for special values of λ , frequencies of different modes, or of their harmonics, can coincide. We plan to investigate in detail the onset of chaos for different configurations of the external trap.

5 Conclusions

In this short paper we have discussed the mean-field equations which describe the collective motion of a trapped weakly-interacting Bose condensate. We have shown by using Poincaré sections that for large energy values the system becomes chaotic. An important question is the following: Can BEC chaotic dynamics be experimentally detected? In our opinion the answer is positive. Nowadays non-destructive images of the dynamics of the condensate can be obtained. The radius of the condensate as a function of time can be detected and its power spectrum analyzed. In fact, one finds the the power spectrum of a chaotic signal is much more complex than for a regular one. Typically, one see few peaks in the regular signal and many peaks surrounded by a lot of noise in a chaotic one. Finally, we observe that various initial conditions for the collective dynamics of the condensate can be obtained by using laser beams or by modulating for a short period the magnetic fields which confine the condensate.

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